Divisibility

Hope Chinese School Fall Week 11

November 4, 2017

1. For any positive integer n, define $\boxed{n}$ to be the sum of the positive factors of n. For example, $\boxed{6} = 1 + 2 + 3 + 6 = 12$. Find $\boxed{\boxed{11}}$.

2. Arrange the digits 3, 4, 5, 8 each exactly once in the blanks such that the largest possible product is made. What is the product?

$$\_\_ \times \_\_$$

3. Find all positive integers n greater than 1 for which $n^2 - 1$ is prime.

4. Find all triples of primes (a, b, c) such that $a + b + c = 20$.

5. (a) Explain why the divisibility rules for 2 and 5 are true.
   (b) Extend your method to prove the divisibility rules for 4 and 8.

6. John is thinking of a two-digit natural number. He reverses its digits and realizes that the new number is smaller. He multiplies it with the original number and gets 574. What is his original number?

7. How many ways are there to select some integers (each at most once) among the integers $1, 2, \cdots, 9$ such that their product has a last digit of 0?

8. $\star$ Think of various pairs of natural numbers a and b and calculate $\gcd(a, b) \times \text{lcm}(a, b)$. What patterns do you notice? Can you prove them?

9. $\star$ There are 24 four-digit whole numbers that use each of the four digits 2, 4, 5 and 7 exactly once. Only one of these four-digit numbers is a multiple of another one. What is it?

10. $\star$ Find the largest prime number that is a divisor of $18! - 16!$.

11. $\star$ Randomtown High has 1000 students and 1000 lockers in a row each labeled from 1 to 1000. One day they decide to do the following:

    The first student opens every single locker (which are initially closed). The second student closes every locker with an even label. The third student goes to each locker with label divisible by 3 and closes it if it's open or opens it if it's closed. This process continues, with the nth student changing the state of every locker with label divisible by n.

    Which lockers remain open at the end, and why? (Hint: try this with 10 students and 10 lockers first.)