Algebraic Manipulations

Hope Chinese School Spring Week 17

January 13, 2018

Problems

1. If $x + y = 6$ and $xy = 4$, find $\frac{1}{x} + \frac{1}{y}$.

2. Given that $x + \frac{1}{x} = 7$, find $x^2 + \frac{1}{x^2}$.

3. Compute the sum of all the roots of $(2x+3)(x-4)+(2x+3)(x-6) = 0$.

4. **Simon's favorite factoring trick:** Solve the following equation in integers a, b:
$$ab + 2a + 3b + 7 = 0.$$

5. Find the sum of all positive x that satisfy
$$x^2 + 3x + \frac{3}{x} + \frac{1}{x^2} = 26.$$

6. Find the value of $\sqrt{2 + \sqrt{2 + \sqrt{2 + \cdots}}}$.

7. Suppose that $4^a = 5$, $5^b = 6$, $6^c = 7$, and $7^d = 8$. What is $a \cdot b \cdot c \cdot d$?

8. If x and y are real numbers for which $(x+y)^2 + (x-y)^2 = 10$ and $(x+y)^4 + (x-y)^4 = 98$, what is the value of xy?

9. A rectangle has area 3 and perimeter 12. Its side lengths are both increased by 1. What is the area of the rectangle now?

10. Let A, M, and C be nonnegative integers such that $A+M+C = 9$. What is the maximum value of $A \cdot M \cdot C + A \cdot M + M \cdot C + C \cdot A$?

11. Simplify $\sqrt{9 + 6\sqrt{2}} + \sqrt{9 - 6\sqrt{2}}$.

12. Evaluate the sum
$$\frac{1}{\sqrt{15}+\sqrt{13}} + \frac{1}{\sqrt{13}+\sqrt{11}} + \frac{1}{\sqrt{11}+3} + \frac{1}{3+\sqrt{7}} + \frac{1}{\sqrt{7}+\sqrt{5}}.$$

13. $\star$ Solve for x:
$$x^3 + 3x^2 + 3x + 7 = 1337.$$

14. $\star$ Let a be a positive real number such that $a^2 + 12a - 1 = 0$. Find the value of $\frac{1}{a+5} - \frac{1}{a+7}$.

15. $\star$ Let a and b be relatively prime integers with $a > b > 0$ and $\frac{a^3-b^3}{(a-b)^3} = \frac{73}{3}$. What is $a - b$?

16. $\star$ Find two four-digit numbers whose product is $4^8 + 6^8 + 9^8$.

17. $\star$ Let a, b, and c be positive integers with $a \geq b \geq c$ such that

$$a^2 - b^2 - c^2 + ab = 2011 \text{ and}$$
$$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997$$

What is a?

Help! I forgot the formulas!

Some useful factorizations and tricks are listed below. But blindly applying them is not going to work here!

- $(x + y)^2 = x^2 + 2xy + y^2$
- $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$. Writing it as $x^3 + y^3 + xy(x + y)$ is often helpful as well.
- $(x + y)(x - y) = x^2 - y^2$
- $(x + 1)(y + 1) = xy + x + y + 1$
- Substitution: abuse this to the fullest extent possible. Heck, abuse everything to the fullest extent possible. It's how you do math.
- Symmetry: add or multiply to "smooth out" many almost-symmetric expressions. Whatever you do, tampering with symmetry is usually not a good idea.
- Actually try something instead of reading this.